PROBLEMS FOR ALGEBRA SECTION

1. Team test

Problem 1. Let A be a finitely generated $\mathbb{Z}$-algebra and let $\mathfrak{m}$ be a maximal ideal of A. Show that $A/\mathfrak{m}$ is a finite field.

Problem 2. Let C be a category. Denote by $1_C : C \to C$ the identity functor on C. A natural transform from 1_C to 1_C consists of a collection $\{\eta_X\}_{X \in Ob(C)}$ such that

- for any $X \in Ob(C)$, η_X is a morphism from X to X;
- for any morphism $f : X \to Y$ in C, the following square is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\eta_Y} & Y. \end{array}$$

We call the set of all natural transforms from 1_C to 1_C the center of C.

(1) Determine the center of the category of abelian groups.
(2) Determine the center of the category of groups.

Problem 3. Let W be the Weyl algebra over a field k, which is the associative algebra generated by $x_1, ..., x_n, y_1, ..., y_n$ such that $[x_i, x_j] = [y_i, y_j] = 0$ and $[x_i, y_j] = \delta^i_j$ for all $1 \le i, j \le n$. ($\delta^i_j = 0$ or 1 is the Kronecker symbol.)

(1) In case $\mathrm{char}(k) = 0$, show that W does not have finite dimensional representation.
(2) In case $\mathrm{char}(k) > 0$, find all finite dimensional representations of W.